Senior problems

S397. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}+\frac{3(ab+bc+ca)}{2(a+b+c)} \geq a+b+c.$$

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Note that

$$\sum_{cyc} \frac{a^2}{a+b}+\frac{3(ab+bc+ca)}{2(a+b+c)} \geq a+b+c \iff \sum_{cyc} \frac{a^2(a+b+c)}{a+b}+\frac{3(ab+bc+ca)}{2} \geq (a+b+c)^2.$$

Since

$$\sum_{cyc} \frac{a^2(a+b+c)}{a+b} = \sum_{cyc} a^2+\sum_{cyc} \frac{a^2 c}{a+b}$$

and by Cauchy-Schwarz Inequality,

$$\sum_{cyc} \frac{a^2 c}{a+b}=\sum_{cyc} \frac{c^2 a^2}{ca+bc} \geq \frac{(ca+ab+bc)^2}{\sum_{cyc}(ca+bc)}=\frac{ab+bc+ca}{2}$$

therefore

$$\sum_{cyc} \frac{a^2(a+b+c)}{a+b}+\frac{3(ab+bc+ca)}{2} \geq a^2+b^2+c^2+\frac{ab+bc+ca}{2}+\frac{3(ab+bc+ca)}{2}=$$

$$a^2+b^2+c^2+2(ab+bc+ca)=(a+b+c)^2.$$

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